DATA-GUIDE QUICK CHART®

ANALYTIC GEOMETRY

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

LEARN & REVIEW IN MINUTES

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1 BASIC CONCEPTS.

A. DIRECTED SEGMENTS, LINES. **1.** If A is the *point initial* and B the *terminal point*, AB becomes the *directed* (or oriented) segment AB. Segment BA has the same length as AB but it is traversed in the opposite sense (from B to A); thus $AB=-BA$. If C is any third point on L, then $AB+BC+CA=0$. **2.** A directed segment on L is *positively* directed when has it *same* sense as L; *negatively* directed when of *opposite* sense. (The signed length of AB is $+$ when positively directed, $-$ when negatively directed.

Fig. 1.

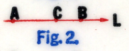
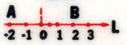


Fig. 2.



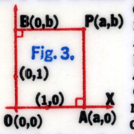


Fig. 3.

B. RECTANGULAR COORDINATE SYSTEM. (Permits location of point in a plane, Fig. 3.) **1.** The coordinates of point A are $(a, 0)$; of B, $(0, b)$. Value a is the *abscissa* or the *x-coordinate* of P; value b is the *ordinate* or *y-coordinate* of P; thus the coordinates of P in the plane are (a,b). **2.** The coordinates of the origin are $(0,0)$. **3.** Thus the abscissa of any point to the *right* of the y-axis is "+"; to the *left*, "−". The ordinate of any point above the x-axis is "+"; below, "−".

2 FUNDAMENTAL RELATIONSHIPS.

A. ANGLES. **1.** *The angle of inclination* (α) of a directed line is angle it makes with directed x-axis, measured in a *ccw* direction ($0 \leqq \alpha < 180°$). **2.** *Slope* (m) of line of inclination α is $\tan\alpha$.

INCLINATION α	0°	acute	obtuse	90°
SLOPE m	0	+	−	∞

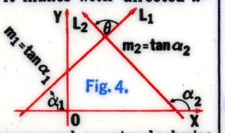
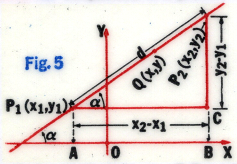


Fig. 4.

3. *The angle between two oriented lines* is the angle θ between their *positive* directions. The angle from L_1 to L_2 is θ measured counterclockwise (positively); the angle from L_2 to L_1 is $180°-\theta$.

B. POINTS AND SEGMENTS. **1.** Projection of a point P on a line L is the foot of the perpendicular dropped from P on to L (Fig. 5). **2.** The projection P_1P_2 on the x-axis is AB, where A is the projection of P_1; B, the projection of P_2. **3.** If the distance between P_1 and P_2 is d (a positive number) and the angle between P_1P_2 and the x-axis is α, then $AB=P_1C=d\cos\alpha$. **4.** Segment P_1P_2 is divided by point Q in ratio $r_1:r_2$ if $P_1Q:QP_2=r_1:r_2$. If $r_1:r_2$ is $+$, Q is an internal point of division; $-$, then Q is an external pt. of division.

Fig. 5



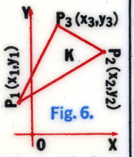


Fig. 6.

x_2-x_1	Projection of P_1P_2 on x-axis.
y_2-y_1	Projection of P_1P_2 on y-axis.
$m=\tan\alpha=\dfrac{y_2-y_1}{x_2-x_1}$	Slope of line P_1P_2. NOTE. When $x_1=x_2$, the line is perpendicular to x-axis and has no slope ($m=\infty$).
$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$	Distance between P_1 and P_2.
$\left(\dfrac{r_2x_1+r_1x_2}{r_1+r_2}, \dfrac{r_2y_1+r_1y_2}{r_1+r_2}\right)$	Point dividing P_1P_2 in ratio $r_1:r_2$.
$\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$	Midpoint of P_1P_2. ($r_1=r_2$).
$\tan\theta=\dfrac{m_2-m_1}{1+m_1m_2}$	Formula for angle from line L_1 having slope m_1, to line L_2 with slope m_2.
$m_1m_2=-1$	Condition that L_1 be perpendicular to L_2.
$m_1=m_2$	Condition that L_1 be parallel to L_2.
$K=\dfrac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$	Area of triangle $P_1P_2P_3$ (Fig. 6).
$K=0$	Condition that points P_1,P_2,P_3 be collinear.

E: Given $P_1(-2, 3)$; P_2 $(4, 6)$. Projection of P_1P_2 on x-axis$=4-(-2)=6$; slope$=(6-3)/[4-(-2)]=1/2$; d between P_1, $P_2=\sqrt{(-2-4)^2+(3-6)^2}=\sqrt{45}$; midpoint of $P_1P_2=\left(\dfrac{-2+4}{2}, \dfrac{3+6}{2}\right)=\left(1, \dfrac{9}{2}\right)$.

3 LOCI AND EQUATIONS.

A. DEFINITIONS. **1.** A locus is a set of points satisfying a given condition or group of conditions. **2.** A curve C is said to be the locus of $f(x,y)=0$ if every point of C has coordinates that satisfy the equation and if every point whose coordinates satisfy the equation lies on C. **E:** The locus of $x^2=1$ is the pair of lines parallel to the y-axis and one unit away from it.

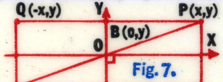
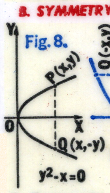


Fig. 7.

B. SYMMETRY. ***1. Symmetry with respect to a line:*** Two points P, Q are symmetric with respect to a line L if L is the perpendicular bisector of segment PQ. The points $P(x,y)$ and $Q(-x,y)$ are symmetric in the y-axis (*Fig.* 7); so are $S(x, -y)$ and $R(-x-y)$. The pts $P(x,y)$ and $S(x,-y)$ are symmetric in x-axis. ***2. Symmetry with respect to a point:*** $P(x,y)$ and $R(-x, -y)$ are symmetric with respect to O (the origin) if O bisects PR.

Fig. 8.



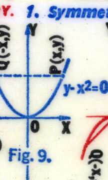


Fig. 9.

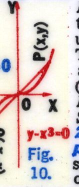


Fig. 10.

3. Symmetry of curve:

IN	CONDITIONS	PROPERTY OF EQUATION	EXAMPLE
X-AXIS	$f(x,-y)=f(x,y)$	Only even powers of y	$y^2-x=0$ (Fig. 8)
Y-AXIS	$f(-x,y)=f(x,y)$	Only even powers of x	$y-x^2=0$ (Fig. 9)
ORIGIN	$f(-x,-y)=\pm f(x,y)$	All terms of odd (or even) degree in x and y	$y-x^3=0$ (Fig. 10) $xy=4$

C. INTERCEPTS OF A CURVE. **1.** *The x-intercepts* are the abscissas of the points of intersection of the curve with the x-axis. **2.** *The y-intercepts* are the ordinates of the points of intersection with the y-axis. **3.** The intercepts are found by letting $y=0$ (or $x=0$) and solving for the real values of x (or y). **E:** x-intercept of $2x-3y=6$ is 3; y-intercept is -2.

D. EXTENT OF A LOCUS. Locus of an equation $f(x,y)=0$ can lie only on those parts of the plane where coordinates x,y that satisfy the equation are both real. **E:** $y=\log x$ exists only where $x>0$. **E:** $y=\sin x$ lies between lines $y=\pm 1$.

E. ASYMPTOTES. An asymptote to a curve is a line, the points of which eventually come closer to curve than any quantity, however small, without crossing curve. **E:** In Fig. 11, the curve $y=2/(x-3)$ exists when $x\neq 3$; thus it has branches on both sides of the line $x=3$, a vertical asymptote since as $x\to 3$, $|y|\to\infty$. x-axis is a horizontal asymptote since as $|x|\to\infty$, $y\to 0$.

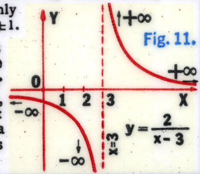


Fig. 11.

4 STRAIGHT LINE.

A. GENERAL EQUATION OF 1ST DEGREE (*in x, y*). The locus of $Ax+By+C=0$, is a straight line and every straight line has an equation of the first degree.

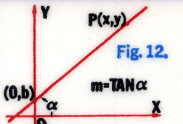


Fig. 12.

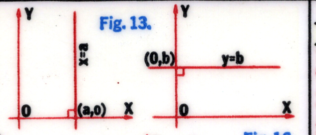
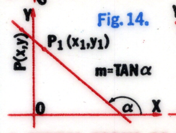


Fig. 13.

Fig. 14.



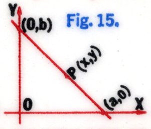


Fig. 15.

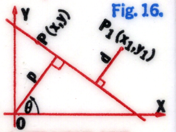


Fig. 16.

B. SPECIAL FORMS OF THE EQUATION OF A STRAIGHT LINE.

$y=mx+b$	Line determined by slope m, y-intercept b. (Fig. 12)
$x=a$	Line parallel to y-axis, $m=\infty$; x-intercept$=a$. (Fig. 13)
$y=b$	Line parallel to x-axis; $m=0$; y-intercept$=b$. (Fig. 13)
$y-y_1=m(x-x_1)$	Line determined by pt. (x_1, y_1), slope m. (Fig. 14)
$(y-y_1)\div(x-x_1)=(y_2-y_1)\div(x_2-x_1)$	Line determined by two pts.
$(x/a)+(y/b)=1$	Two-intercept form; $a\neq 0$, $b\neq 0$. (Fig. 15)
$x\cos\theta+y\sin\theta-p=0$; $p>0$	Normal form: Line determined by length (p) of normal from origin and angle to x-axis. (Fig. 16)
$\dfrac{A}{\pm\sqrt{A^2+B^2}}x+\dfrac{B}{\pm\sqrt{A^2+B^2}}y+\dfrac{C}{\pm\sqrt{A^2+B^2}}=0$	Reduction of Gen. Eq. to normal form.

(Sign before radical is chosen opposite to C. If $C=0$, use sign of B.)

E: Line $3x+4y+10=0$ has normal form $(3/-5)x+(4/-5)y+10/-5=0$. Its distance from the origin is $p=10/5=2$. Since $y=-(3/4)x-5/2$, the line has the slope $m=-3/4$ and y-intercept $b=-5/2$.

C. PARALLEL AND PERPENDICULAR LINES. Given lines L_1, L_2. (*Fig.* 17)

Eq. of L_1	$A_1x+B_1y+C_1=0$	$y=m_1x+b_1$
Eq. of L_2	$A_2x+B_2y+C_2=0$	$y=m_2x+b_2$
L_1 and L_2 coincide	$\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}$	$m_1=m_2$ $b_1=b_2$
L_1 and L_2 parallel	$\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}\neq\dfrac{C_1}{C_2}$	$m_1=m_2$ $b_1\neq b_2$
L_1 and L_2 rt. angles	$A_1A_2+B_1B_2=0$	$m_1m_2=-1$

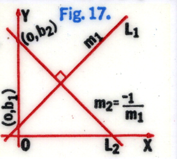


Fig. 17.

E: Equation of line passing thru point $(4, -1)$ and perpendicular to line $y=2x-3$. **Sol.** Slope of required line is $-1/2$; equation is $y+1=-1/2(x-4)$ or $x+2y-2=0$.

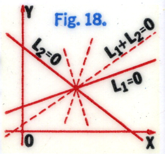


Fig. 18.

D. DISTANCE OF A POINT TO A LINE. (Fig. 16)

EQUATION OF L	DISTANCE OF $P_1(x_1,y_1)$
$Ax+By+C=0$	$d=\dfrac{\lvert Ax_1+By_1+C\rvert}{\sqrt{A^2+B^2}}$
$x\cos\theta+y\sin\theta-p=0$	$d=\lvert x_1\cos\theta+y_1\sin\theta-p\rvert$

E. FAMILIES OF STRAIGHT LINES. ***1. Equation L:*** $y=mx+b$ represents an infinite number of straight lines each having a specific value of m and b. **2.** If value of m is held fixed and b is allowed to vary, the equation represents a family of parallel lines of slope m. **3.** If b is fixed and m varies, equation is a family of lines passing through $(0,b)$. **4.** Both b and m are *parameters*; to each corresponds one line of the family. ***5. Pencil of lines:*** Given two lines, $L_1=A_1x+B_1y+C_1=0$ and $L_2=A_2x+B_2y+C_2=0$, a pencil of straight lines is given by $L_1+kL_2=0$. (Fig. 18)

CONDITION	FAMILY	Parameter k
L_1, L_2 intersect	All lines pass thru pt. of intersection	To every real value of k corresponds 1 line of pencil. To $k=0$, corresponds $L_1=0$, to $k=\infty$, corresponds $L_2=0$.
L_1, L_2 parallel	Pencil of parallel lines.	

E: Find equation of line determined by the point $(1,0)$ and the point of intersection of line $x+y-3=0$, $x-y+1=0$. **Sol.** Required line belongs to pencil $(x+y-3)+k(x-y+1)=0$. Since $(1,0)$ lies on line, $(1+0-3)+k(1-0+1)=0$, $k=1$; thus $(x+y-3)+(x-y+1)=0$ or $x-1=0$.

5 CONIC SECTIONS — CIRCLE.

A. DEF. *A circle is locus of a point* (P) *which moves such that it is always a constant distance (the radius,* r*) from a fixed point [center,* C *at* (h, k)].

EQUATION	CENTER	RADIUS
$(x-h)^2+(y-k)^2=r^2$	(h,k)	r
$x^2+y^2+dx+ey+f=0$	$\left(-\dfrac{d}{2}, -\dfrac{e}{2}\right)$	$\dfrac{\sqrt{d^2+e^2-4f}}{2}$

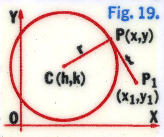


Fig. 19.

NOTE: When $d^2+e^2-4f=0$, circle is a point $(-d/2, -e/2)$ and $r=0$. When $d^2+e^2-4f<0$, circle is imaginary.

E: Find the equation of the circle with center at $(2, 1)$ that is tangent to the line $3x+4y-5=0$. **Sol.** Radius = distance of $(2, 1)$ from the line or $\lvert 3(2)+4(1)-5\rvert\div\sqrt{3^2+4^2}=1$. Ans: $(x-2)^2+(y-1)^2=1$.

B. TANGENT TO A CIRCLE. Length of a tangent from $P_1(x_1, y_1)$ to a circle is $\sqrt{(x_1-h)^2+(y_1-k)^2-r^2}$. Equation of tangent when $P_1(x_1, y_1)$ is on the circle is $(x_1-h)(x-h)+(y_1-k)(y-k)=r^2$.

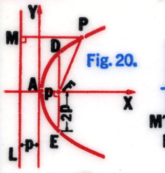
C. PENCIL OF CIRCLES. **1.** Two circles determine a pencil of circles. Let C_1 $(x^2+y^2+d_1x+e_1y+f_1=0)$ and C_2 $(x^2+y^2+d_2x+e_2y+f_2=0)$ be two given circles. Pencil of circles they determine is given by $(x^2+y^2+d_1x+e_1y+f_1)+k(x^2+y^2+d_2x+e_2y+f_2)=0$; $C_1+kC_2=0$, where k is parameter. **2.** To each value of k (except to $k=-1$) corresponds a circle of pencil. By convention, C_2 corresponds to $k=\infty$ (not a number). To $k=-1$, corresponds a line L called radical axis of pencil. **NOTE:** when C_1 and C_2 are concentric, no radical axis. **3.** L is locus of all points from which tangent line segments drawn to C_1 and C_2 have equal length. L is perpendicular to line of centers of C_1 and C_2. **4. Types:** C_1, C_2 intersect in two points (*radical axis is the common chord*), are tangent (*radical axis is the common tangent*), do not intersect.

6 CONIC SECTIONS—PARABOLA, ELLIPSE, HYPERBOLA

A. DEF. **1.** *These conic sections are locus of a point* (P) *which moves such that its distance from a fixed point (a focus,* F*) is in a constant ratio (eccentricity,* e*) to its distance from a fixed line (a directrix,* L*).* Conic sections are the curves of intersection of a circular cone by a plane. **2. The principal axis** is the line through a focus perpendicular to a directrix. **3. A vertex** (A; A′) is a point at which principal axis intersects curve. **4. The center** (C) is mid-point of the segment which joins the foci of the ellipse or hyperbola. **5. The major axis** (AA′ *in the ellipse*) or **the transverse axis** (AA′ *in the hyperbola*) is segment of the principal axis that connects vertices of the conic. **6. The minor axis** (BB′ *in ellipse*) or **the conjugate axis** (BB′ *in hyperbola*) is the segment of a line through the center, perpendicular to principal axis of the conic section. **7. A latus rectum** (DE; D′E′) is a chord through a focus, perpendicular to principal axis, intersected by curve. **8. A focal radius** (FP; F′P′) is any line segment from a focus to any point on curve.

B. IDENTIFICATION OF THE PARABOLA, ELLIPSE, AND HYPERBOLA

PARABOLA	ELLIPSE	HYPERBOLA
The locus of a point (P) *whose distance from a fixed line* (L) *always equals its distance from a fixed point* (F) *not on* L.	*The locus of a point* (P) *such that the sum of its distances from two fixed points* (F, F′) *is a constant.*	*The locus of a point* (P) *such that the difference of its distances from two fixed points* (F, F′) *is a constant.*



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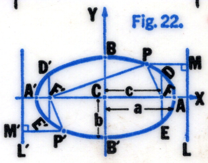


Fig. 22.

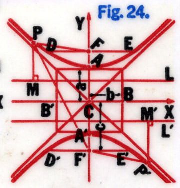
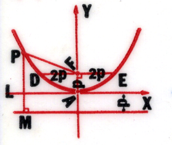


Fig. 24.



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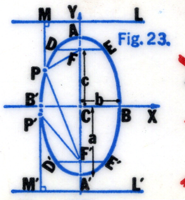

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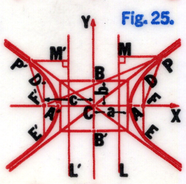


Fig. 25.

	PARABOLA		ELLIPSE		HYPERBOLA	
	$\dfrac{FP}{MP}=e=1$; $FP=MP$		$\dfrac{FP}{MP}=\dfrac{F'P'}{M'P'}=e<1$ $\lvert FP+F'P\rvert=2a$		$\dfrac{FP}{MP}=\dfrac{F'P'}{M'P'}=e>1$ $\lvert FP-F'P\rvert=2a$	
	On the Principal axis: $AF=p$		*Major axis:* $AA'=2a$ *Minor axis:* $BB'=2b$ $FF'=2c$ $a^2=b^2+c^2$		*Transverse axis:* $AA'=2a$ *Conjugate axis:* $BB'=2b$ $FF'=2c$ $c^2=a^2+b^2$	
	PRINCIPAL AXIS		MAJOR AXIS		TRANSVERSE AXIS	
	x-axis	y-axis	x-axis	y-axis	x-axis	y-axis
EQ	$y^2=4px$	$x^2=4py$	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	$\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$
C	—	—	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
F	$(p, 0)$	$(0, p)$	$(c, 0)$	$(0, c)$	$(c, 0)$	$(0, c)$
F′	—	—	$(-c, 0)$	$(0, -c)$	$(-c, 0)$	$(0, -c)$
A	$(0, 0)$	$(0, 0)$	$(a, 0)$	$(0, a)$	$(a, 0)$	$(0, a)$
A′	—	—	$(-a, 0)$	$(0, -a)$	$(-a, 0)$	$(0, -a)$
B	—	—	$(0, b)$	$(b, 0)$	$(0, b)$	$(b, 0)$
B′	—	—	$(0, -b)$	$(-b, 0)$	$(0, -b)$	$(-b, 0)$
DE D′E′	$\lvert 4p\rvert$	$\lvert 4p\rvert$	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$	$\dfrac{2b^2}{a}$
e	1	1	$(c/a)<1$	$(c/a)<1$	$(c/a)>1$	$(c/a)>1$
L	$x=-p$	$y=-p$	$x=a/e$	$y=a/e$	$x=a/e$	$y=a/e$
L′	—	—	$x=-a/e$	$y=-a/e$	$x=-a/e$	$y=-a/e$
ASY	—	—	—	—	$y=\pm(b/a)x$	$y=\pm(a/b)x$
TAN AT X_1, Y_1	$yy_1=2p(x+x_1)$ $xx_1=2p(y+y_1)$		$\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$	$\dfrac{xx_1}{b^2}+\dfrac{yy_1}{a^2}=1$	$\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$	$\dfrac{yy_1}{a^2}-\dfrac{xx_1}{b^2}=1$

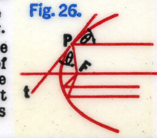
E1: For the ellipse $9x^2+25y^2=225$, $(x^2/25)+(y^2/9)=1$; $a^2=25$, $b^2=9$, $c^2=a^2-b^2=25-9=16$; $e=c/a=4/5$; the foci are $(\pm 4, 0)$; the directrices are $x=\pm a/e=\pm 25/4$.

E2: One focus of a hyperbola is at $(0, 5)$, its center is at origin, eccentricity is $5/3$. Find the equation. **Sol.** $c=5$; $e=5/3=5/a$, thus $a=3$; $b^2=c^2-a^2=25-9=16$; the foci are on the y-axis: $(y^2/9)-(x^2/16)=1$.

E3: Find equation of parabola that has vertex at the origin and directrix $x=4$. **Sol.** $p=-4$, thus $y^2=-16x$.

C. OPTICAL PROPERTIES OF CONIC SECTIONS. **1.** The focal radius FP of a parabola at any point P of the curve and the parallel line to axis at P make equal angles with the tangent at P. **2.** A tangent to the ellipse and hyperbola at any point makes equal angles with the focal radii of that point.

Fig. 26.



7 TRANSFORMATION OF COORDINATES.

A. TRANSLATION OF AXES. **1.** Given two sets of axes, the old xy-axes with origin at O; the new or or $x'y'$-axes with origin at O', so that the x'- and y'-axes are parallel respectively to the x- and y-axes. Either set of axes is said to be translated with respect to the other. **2.** Let O' have the coordinates (h,k) with respect to the x,y-axes; then a point P has two sets of coordinates: (x,y) and (x',y') ***3. Equations of translation:*** $x=x'+h$, $y=y'+k$; $x'=x-h$, $y'=y-k$

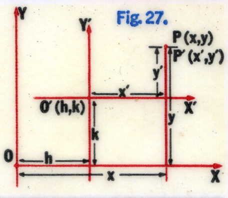


Fig. 27.

	PARABOLA		ELLIPSE		HYPERBOLA	
	PRINCIPAL AXIS ∥ TO x-axis	y-axis	MAJOR AXIS ∥ TO x-axis	y-axis	TRANSVERSE AXIS ∥ TO x-axis	y-axis
EQ	$(y-k)^2=4p(x-h)$		$(x-h)^2/a^2+(y-k)^2/b^2=1$		$(x-h)^2/a^2$ $(y-k)^2/b^2=1$	
	$(x-h)^2=4p(y-k)$		$(x-h)^2/b^2+(y-k)^2/a^2=1$		$(y-k)^2/a^2-(x-h)^2/b^2=1$	
C	—	—	(h, k)	(h, k)	(h, k)	(h, k)
A	(h, k)	(h, k)	$(h+a, k)$	$(h, k+a)$	$(h+a, k)$	$(h, k+a)$
A′	—	—	$(h-a, k)$	$(h, k-a)$	$(h-a, k)$	$(h, k-a)$
F	$(h+p, k)$	$(h, k+p)$	$(h+c, k)$	$(h, k+c)$	$(h+c, k)$	$(h, k+c)$
F′	—	—	$(h-c, k)$	$(h, k-c)$	$(h-c, k)$	$(h, k-c)$
B	—	—	$(h, k+b)$	$(h+b, k)$	$(h, k+b)$	$(h+b, k)$
B′	—	—	$(h, k-b)$	$(h-b, k)$	$(h, k-b)$	$(h-b, k)$
L	$x=h-p$	$y=k-p$	$x=h+a/e$	$y=k+a/e$	$x=h+a/e$	$y=k+a/e$
L′	—	—	$x=h-a/e$	$y=k-a/e$	$x=h-a/e$	$y=k-a/e$
ASY	—	—	—	—	$y-k=\pm\dfrac{b}{a}(x-h)$	$y-k=\pm\dfrac{a}{b}(x-h)$

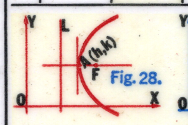


Fig. 28.

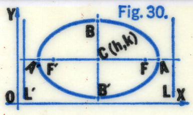


Fig. 30.

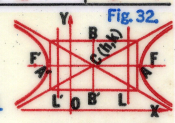


Fig. 32.

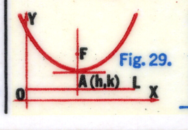


Fig. 29.

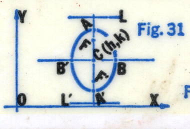


Fig. 31

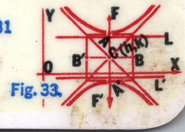


Fig. 33.

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QUICK CHARTS PUT THE FACTS IN FULL VIEW:
PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.
QUICK CHARTS HELP YOU REMEMBER:
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E1: Find the equation of curve $y^2-2y-4x+9=0$ referred to parallel axes with the new origin at (2,1). **Sol.** $h=2$, $k=1$; $x=x'+2$, $y=y'+1$. Substitute: $(y'+1)^2-2(y'+1)-4(x'+2)+9=0$. Simplify: $y'^2=4x'$, which is a parabola in standard form with respect to tne $x'y'$-axes. **E2:** By means of a translation of axes, transform $9x^2+4y^2+18x-24y+9=0$ into equation without terms of the 1st degree. **Sol.** Substitute $x=x'+h$ and $y=y'+k$, thus $9x'^2+4y'^2+(18h+18)x'+(8k-24)y'+(9h^2+4k^2+18h-24k+9)=0$. Equating to zero the coefficients of x' and y' yields $18h+18=0$ and $8k-24=0$, thus $h=-1$, $k=3$ and the transformed equation is $9x'^2+4y'^2=36$; an ellipse in standard form with respect to the $x'y'$-axes.

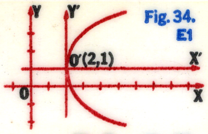

Fig. 34. E1

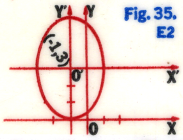

Fig. 35. E2

B. ROTATION OF AXES AROUND ORIGIN. **1.** Let axes of coordinates be rotated around the origin O through angle θ. (Fig. 36) Let the point P, whose coordinates are (x,y) with respect to the xy-axes, have the coordinates (x',y') with respect to the rotated $x'y'$-axes.

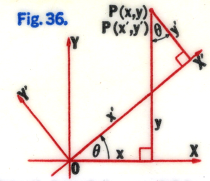

Fig. 36.

2. Equations of rotation:

$$x=x'\cos\theta-y'\sin\theta$$
$$y=x'\sin\theta+y'\cos\theta$$
$$x'=x\cos\theta+y\sin\theta$$
$$y'=-x\sin\theta+y\cos\theta$$

E: Transform the equation of the equilateral hyperbola $xy=2a^2$ by rotating the axes through an angle of 45°. **Sol.** $x=x'\cos 45°-y'\sin 45°=(x'-y')/\sqrt{2}$; $y=x'\sin 45°+y'\cos 45°=(x'+y')/\sqrt{2}$, thus

$$xy=\frac{(x'-y')(x'+y')}{(\sqrt{2})^2}=\frac{x'^2-y'^2}{2}=2a^2,\text{ or } x'^2-y'^2=4a^2$$

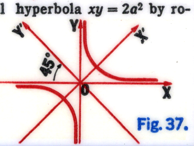

Fig. 37.

8 GENERAL EQUATION OF THE SECOND DEGREE

A. DEFINITIONS. **1.** Every conic (section) has an equation of the 2nd degree in rectangular coordinates. General equation of 2nd degree is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ (A,B,C not all zero). The locus (when it exists) is always a conic. ***2. Proper conics:*** circle, parabola, ellipse, hyperbola. ***3. Improper conics:*** a pair of straight lines (which may coincide) and a single point. **E:** The locus of $x^2-y^2=0$ is the pair of straight lines $x+y=0$ and $x-y=0$. The locus of $x^2=0$ is the y-axis counted twice; locus of $2x^2+y^2=0$ is origin; equation $x^2+y^2=-1$ has no locus.

B. CONICS WITH PRINCIPAL AXIS PARALLEL TO AN AXIS OF COORDINATES. The xy-term in its equation is missing ($B=0$) and vice versa: $Ax^2+Cy^2+Dx+Ey+F=0$.

CIRCLE: $A=C\neq 0$	**PARABOLA:** $A=0, C\neq 0; C=0, A\neq 0$
ELLIPSE: $AC>0$	**HYPERBOLA:** $AC<0$

E: By means of a translation of axes, find the standard form of the equation of the conic $9x^2+4y^2+18x-16y=11$. **Sol.** Collecting separately the terms in x and y, $9(x^2+2x+\quad)+4(y^2-4y+\quad)=11$, then completing the squares within the parentheses gives $9(x^2+2x+1)+4(y^2-4y+4)=11+9+16=36$ or $9(x+1)^2+4(y-2)^2=36$. The translation $x'=x+1$, $y'=y-2$ yields $9x'^2+4y'^2=36$ or $(x'^2/4)+(y'^2/9)=1$; an ellipse with $a=3$ and $b=2$.

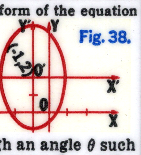

Fig. 38.

C. TO REMOVE THE xy-TERM. Rotate the axes through an angle θ such that $\cot 2\theta=(A-C)/B$ or $\tan 2\theta=B/(A-C)$. **E:** By means of a rotation remove the xy-term from $8x^2+4xy+5y^2=9$. **Sol.** $\tan 2\theta=4/(8-5)=4/3$; $\cos 2\theta=\frac{3}{5}$; $\sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-(3/5)}{2}}=\frac{1}{\sqrt{5}}$; $\cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}=\frac{2}{\sqrt{5}}$; the required rotation is given by $x=x'\cos\theta-y'\sin\theta=(2x'-y')/\sqrt{5}$; $y=x'\sin\theta+y'\cos\theta=(x'+2y')/\sqrt{5}$. Substituting for x and y in the equation gives $\frac{8(2x'-y')^2}{5}+\frac{4(2x'-y')(x'+2y')}{5}+\frac{5(x'+2y')^2}{5}=9x'^2+4y'^2=9$ (ellipse, $a=3/2$, $b=1$).

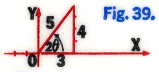

Fig. 39.

9 INVARIANTS UNDER DISPLACEMENTS OF THE AXES.

A. DEFINITIONS. **1.** When the axes are displaced (*translated*, *rotated*, or *both*), the general equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is transformed into $A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0$. **2.** A function of the coefficients, $I=f(A,B,C,\ldots,F)$ such that $f(A,B,C,\ldots,F)=f(A',B',C',\ldots F')$, is an invariant under displacements of the axes.

B. INVARIANTS. B^2-4AC; $\Delta=4ACF-B^2F-AE^2-CD^2+BDE$

C. CLASSIFICATION OF THE LOCI OF GENERAL EQUATION OF 2ND DEGREE BY MEANS OF B^2-4AC AND Δ.

	$B^2-4AC<0$	$B^2-4AC=0$	$B^2-4AC>0$
$\Delta\neq 0$	Ellipse if $A\Delta<0$ No locus if $A\Delta>0$	Parabola	Hyperbola
$\Delta=0$	Point	Two parallel lines, two coincident lines, or no locus	Two intersecting lines

E: Identify the curve $8x^2+4xy+5y^2-9=0$. **Sol.** $B^2-4AC=16-4(8)(5)<0$; $A\Delta=8[4(8)(5)(-9)-(16)(-9)+0+0+0]<0$; curve is an ellipse.

10 POLAR COORDINATES. A. DEFINITIONS.

1. *Position of point* **P** *is determined by its distance r from a fixed point* **0** *and the angle θ that* **OP** *makes with a fixed indefinite line* **OA** *(the initial line.)* **2.** The ordered pair of numbers (r,θ) are called the *polar coordinates* of P; r is the *radius vector* of P and θ its *vectorial angle*. The rays eminating from the pole O are described by their vectorial angles; OC is the 90° ray. **NOTE:** (r,θ), $(r,\theta+360°)$, $(-r,\theta+180°)$ represent same point.

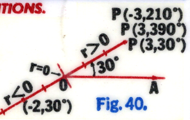

Fig. 40.

B. TRANSFORMATION OF POLAR AND RECTANGULAR COORDINATES.

1. $x=r\cos\theta$ $\quad$ $y=r\sin\theta$
2. $x^2+y^2=r^2$ $\quad$ $y/x=\tan\theta$

E1: Transform to polar coordinates the equation of the circle $x^2+y^2-2ax=0$. **Sol.** $x^2+y^2-2ax=r^2-2ar\cos\theta=0$; $r=2a\cos\theta$. **E2:** Transform to rectangular coordinates the equation of the lemniscate $r^2=\sin 2\theta$. **Sol.** $r^2=2\sin\theta\cos\theta$; $r^4=2r\sin\theta\cdot r\cos\theta$ and $(x^2+y^2)^2=2yx$.

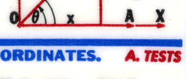

Fig. 41.

11 PLOTTING CURVES IN POLAR COORDINATES. A. TESTS FOR SYMMETRY.

1. $P(r,\theta)$; $Q(r,-\theta)$ are symmetric in *initial line* or *polar axis*. **2.** $P(r,\theta)$; $S(r,180°-\theta)$ are symmetric in the *90° line*. **3.** $P(r,\theta)$; $T(-r,\theta)$ are symmetric in *pole*. ***4. To test a curve $r=f(\theta)$ for symmetry in the polar axis,*** substitute $-\theta$ for θ; if $f(-\theta)=f(\theta)$ the curve is symmetric. **E:** $r=1+\cos\theta$ is symmetric in the polar axis since $f(-\theta)=1+\cos(-\theta)=1+\cos\theta=f(\theta)$. ***5. Test for symmetry in the 90° line:*** Substitute $180°-\theta$ for θ. There is symmetry if $f(180°-\theta)=f(\theta)$. $r=1+\sin\theta$; $f(180°-\theta)=1+\sin(180°-\theta)=1+\sin\theta=f(\theta)$. ***6. Test for symmetry in the pole:*** Substitute $-r$ for r; if eq. unchanged, there is symmetry.

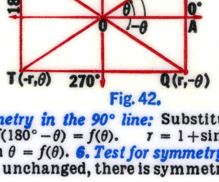

Fig. 42.

B. TESTS FOR EXTENT. **1.** When for all values of θ, $|f(\theta)|<k$ a positive number, a curve $r=f(\theta)$ lies within a circle of radius k and center at pole. **2.** When in the sector $\alpha<\theta<\beta$, r is imaginary, no points of the curve $r=f(\theta)$ lie in that sector. **E:** Plot lemniscate $r^2=4\cos 2\theta$. **Sol.** Since $\cos(-2\theta)=\cos 2\theta$, the curve is symmetric in polar axis. Since $(-r)^2=r^2$ it is symmetric in pole. Since $\cos 2\theta$ is negative when $45°<\theta<135°$ and $225°<\theta<315°$, there is no locus in these sectors. Since $|r|\leq 2$, the curve lies in circle of radius $=2$.

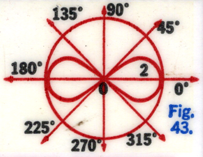

Fig. 43.

C. STRAIGHT LINE (POLAR AND RECTANGULAR COORDINATES).

GENERAL CASE $\qquad$ **SPECIAL CASES**

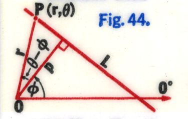

Fig. 44.

$r\cos(\theta-\phi)=p$

$x\cos\phi+y\sin\phi-p=0$

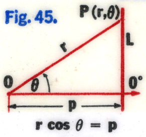

Fig. 45.

$r\cos\theta=p$

$x=p$

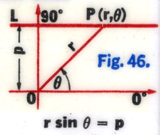

Fig. 46.

$r\sin\theta=p$

$y=p$

D. POLAR EQUATION OF CIRCLE. **NOTE:** R is radius.

EQUATION $\qquad$ **SPECIAL CASES**

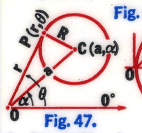

Fig. 47.

$r^2-2ar\cos(\theta-\alpha)+a^2=R^2$

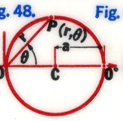

Fig. 48.

$r=2a\cos\theta$

$x^2+y^2=2ax$

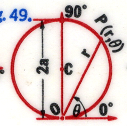

Fig. 49.

$r=2a\sin\theta$

$x^2+y^2=2ay$

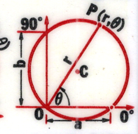

Fig. 50.

$r=a\cos\theta+b\sin\theta$

$x^2+y^2=ax+by$

E. POLAR EQUATION OF PARABOLA, ELLIPSE, HYPERBOLA. **1.** Equation in polar coordinates is $r=ep/(1\pm e\cos\theta)$; **e** is the eccentricity, **p** the distance from directrix **L** to focus **F**. The pole is at F and the directrix is perpendicular to the initial ray. **2.** When directrix is parallel to the initial ray, the equation of the conic takes the form $r=ep/(1\pm e\sin\theta)$.

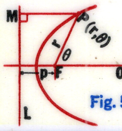

Fig. 51.

12 PARAMETRIC EQUATIONS. A. DEFINITION.

1. Rectangular coordinates of points on a curve C can be expressed as functions of an auxiliary variable: $x=f(t)$, $y=g(t)$. The variable t is called *parameter*. **2.** When parameter is eliminated, we obtain the single equation $F(x,y)=0$ of C. **E:** The parametric equations $x=t+1$, $y=2t$ represent the line $y=2(x-1)$.

B. PARAMETRIC EQUATIONS.

1. LINE	2. CIRCLE	3. ELLIPSE	4. HYPERBOLA
$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$	$x^2+y^2=a^2$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
$x=x_1+t(x_2-x_1)$ $y=y_1+t(y_2-y_1)$	$x=a\cos\theta$ $y=a\sin\theta$	$x=a\cos\theta$ $y=b\sin\theta$	$x=a\sec\theta$ $y=b\tan\theta$

13 SOLID ANALYTIC. A. DEFINITIONS.

1. If two lines are *coplanar* they either *intersect* or are *parallel*; if not, they are "*skew.*" **2.** The angle between two directed coplanar lines is the angle between their *positive* directions. **3.** The angle between two oriented skew lines L_1 and L_2 is the angle between two intersecting oriented lines L'_1 and L'_2 such that $L'_1 \parallel L_1$ and $L'_2 \parallel L_2$. **4.** The projection of P on L is the foot A of the perpendicular from P onto L. A is also intersection of plane through P perpendicular to L. The projection of directed segment PQ on L is AB. **5.** Sum of projections of PQ and QR equals projection of PR. **6.** $AB=PQ\cos\theta$; AB is "+" when θ is <90°; "−" when θ>90°.

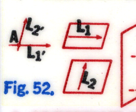

Fig. 52.

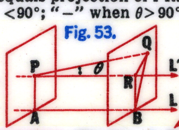

Fig. 53.

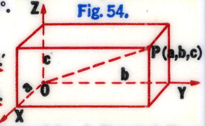

Fig. 54.

B. COORDINATE SYSTEM. (Fig. 54) **1.** Formed by 3 directed lines x, y, z through a point O (origin), each line perpendicular to other two. **2.** The coordinates of a point P in space is the ordered triad of numbers (a,b,c) if a,b,c are the projections of OP on x-, y-, z-axes respectively. **3.** *Coordinate planes*, determined by pairs of coordinate axes, divide space into *8 octants* (coordinates are all "+" in the 1st octant).

C. DIRECTION COSINES AND NUMBERS. **1.** If oriented line L thru O makes angles α, β, γ with x, y, and z axes (Fig. 55) then α,β,γ, are *direction angles* of L and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the *direction cosines* of L. **2.** Direction angles of a directed line L' (not through O) are same as those of the directed line L parallel to L' through O. **3.** When sense of L is reversed, its direction angles are changed into their *supplements* and signs of directed cosines of L *reversed* **NOTE:** The direction numbers of L are any three numbers l, m, n, proportional to direction cosines of line L. Any three numbers (at least one of which $\neq 0$) can be direction numbers. The projections of any directed segment on L are direction numbers of L.

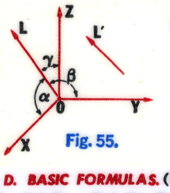

Fig. 55.

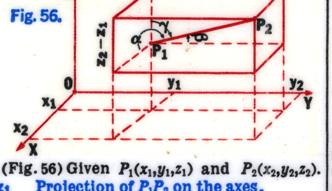

Fig. 56.

D. BASIC FORMULAS. (Fig. 56) Given $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

x_2-x_1; y_2-y_1; z_2-z_1 — Projection of P_1P_2 on the axes.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ — Distance between P_1 and P_2

$x=\frac{r_2x_1+r_1x_2}{r_1+r_2}$; $y=\frac{r_2y_1+r_1y_2}{r_1+r_2}$; $z=\frac{r_2z_1+r_1z_2}{r_1+r_2}$ — Point that divides P_1P_2 in ratio $r_1{:}r_2$.

$x=\frac{x_1+x_2}{2}$; $y=\frac{y_1+y_2}{2}$; $z=\frac{z_1+z_2}{2}$ — Midpoint formula

$\cos\alpha=\frac{x_2-x_1}{d}$; $\cos\beta=\frac{y_2-y_1}{d}$; $\cos\gamma=\frac{z_2-z_1}{d}$ — Direction cosines of P_1P_2.

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ — Direction cosines formula.

$\cos\alpha=\frac{l}{\pm\sqrt{l^2+m^2+n^2}}$; $\cos\beta=\frac{m}{\pm\sqrt{l^2+m^2+n^2}}$; $\cos\gamma=\frac{n}{\pm\sqrt{l^2+m^2+n^2}}$ — Direction cosines of line with the direction numbers l,m,n.

$\cos\theta=\frac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\,\sqrt{l_2^2+m_2^2+n_2^2}}$ — Acute angle between two lines with direction no's. l_1,m_1,n_1 and l_2,m_2,n_2.

$\cos\theta=\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2$ — Angle between two directed lines.

$l_1{:}m_1{:}n_1=l_2{:}m_2{:}n_2$ — Condition that 2 lines be parallel.

$l_1l_2+m_1m_2+n_1n_2=0$ — Condition that 2 lines be perpendicular.

$l{:}m{:}n=\begin{vmatrix}m_1 & n_1\\ m_2 & n_2\end{vmatrix}:\begin{vmatrix}n_1 & l_1\\ n_2 & l_2\end{vmatrix}:\begin{vmatrix}l_1 & m_1\\ l_2 & m_2\end{vmatrix}$ — Direction numbers of line perpendicular to directions l_1,m_1,n_1 and l_2,m_2,n_2.

14 SURFACES AND CURVES. A. BASIC CONCEPTS.

1. ***A surface*** is locus of an equation in x,y,z if it contains every point whose coordinates satisfy the equation, and only those. **2.** The intersection of a *surface* with a *coordinate plane* is its ***trace*** on that plane. **3.** The intersection of a *surface* with an *axis* are its ***intercepts*** on the axis. **E:** The locus of $x+2y=2$ is a plane parallel to the z-axis. Its trace on the xy-plane is a line whose equations are $x+2y=2$ and $z=0$. Its trace on the yz-plane is a line parallel to the z-axis. Its intercepts on the x-axis is 2; on the y-axis it is 1; no z-intercept (Fig. 57). **4.** A curve in space can be represented by the equations of *two surfaces* that contain it as their complete intersection. Sometimes *three* equations are necessary: *when two surfaces intersect in more than one curve.* **5.** A curve can also be represented *parametrically:* $x=f(t)$, $y=g(t)$, $z=h(t)$.

E1: $x^2+y^2+z^2=25$ is equation of sphere of radius 5 with center at origin.
E2: $x^2+y^2+z^2=25$, $z=3$ represent circle formed by intersection of sphere of radius 5 by a plane parallel to the xy-plane and 3 units above it.

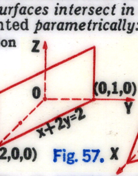

Fig. 57.

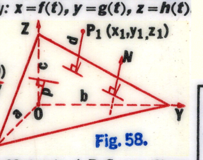

Fig. 58.

B. FORMULAS FOR PLANES Fig. 58. ***NOTE 1:*** Coefficients A,B,C, are direction numbers of any normal to the plane; $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are the direction cosines of the normal. ***NOTE 2:*** Plane S_1 is $A_1x+B_1y+C_1z+D_1=0$ and S_2 is $A_2x+B_2y+C_2z+D_2=0$.

$Ax+By+Cz+D=0$ — General form (every plane has an equation of the first degree, and vice versa).

$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0$ — Normal form (length of normal from origin is p).

$\frac{Ax+By+Cz+D}{\pm\sqrt{A^2+B^2+C^2}}=0$ — Transformation of general equation to normal form.

$l(x-x_1)+m(y-y_1)+n(z-z_1)=0$ — Plane determined by point (x_1,y_1,z_1), normal direction l,m,n.

$(x/a)+(y/b)+(z/c)=1$ — Intercept form (3 non-zero intercepts).

$Ax+By+D=0$ — Plane parallel to z-axis.

$z=k$ — Plane parallel to xy-plane.

$\cos\theta=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\,\sqrt{A_2^2+B_2^2+C_2^2}}$ — Acute angle between S_1 and S_2.

$A_1{:}B_1{:}C_1=A_2{:}B_2{:}C_2$ — Condition that S_1 be parallel to S_2.

$A_1{:}B_1{:}C_1{:}D_1=A_2{:}B_2{:}C_2{:}D_2$ — Condition that S_1,S_2 coincide.

$A_1A_2+B_1B_2+C_1C_2=0$ — Condition that S_1 be perpendicular to S_2.

$(A_1x+B_1y+C_1z+D_1)+k(A_2x+B_2y+C_2z+D_2)=0$ — Pencil of planes determined by S_1 and S_2.

$\frac{|A_1x_1+B_1y_1+C_1z_1+D_1|}{\sqrt{A_1^2+B_1^2+C_1^2}}$ — Distance from S_1 to $P_1(x_1,y_1,z_1)$.

$d=|x_1\cos\alpha+y_1\cos\beta+z_1\cos\gamma-p|$ — Dist. from plane to $P_1(x_1,y_1,z_1)$.

E: Find the distance of $P(3,1,-2)$ from the plane $2x+y+2z-6=0$. **Sol.** Evaluate the normal form: $(2x+y+2z-6)\div(\sqrt{2^2+1^2+2^2})=|2(3)+1(1)+2(-2)-6|\div 3=1$.

C. STANDARD FORMS FOR EQ. OF A STRAIGHT LINE.

$\begin{cases}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{cases}$ — Line determined by intersection of two planes.

$x=az+b$; $y=cz+d$; Line not parallel to xy-plane (projection form).

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ — Line determined by point (x_1,y_1,z_1); direction numbers (l,m,n) (Symmetric form).

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ — Two point form for line thru $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

$x=a_1t+b_1$; $y=a_2t+b_2$; $z=a_3t+b_3$ — Parametric form for line thru pt. (b_1,b_2,b_3) with direction numbers a_1, a_2, a_3.

E: Find equations of the line thru $P(1,2,-1)$ and perpendicular to plane $4x+2y-3z+1=0$. **Sol.** The direction numbers are 4, 2, −3 giving $(x-1)/4=(y-2)/2=(z+1)/-3$.

15 QUADRIC SURFACES. A. GENERAL EQUATION OF THE 2ND DEGREE:

$ax^2+by^2+cz^2+dxy+eyz+fxz+gx+hy+lz+m=0$. **1.** A ***quadric surface*** has a 2nd deg. equation. **2.** When *left side* is factorable into two linear factors, the quadric is *improper* (degenerate) and is composed of a pair of planes. **E:** $x^2-y^2=0$ represents the planes $x-y=0$ and $x+y=0$. **NOTE:** A 2nd degree equation may have no locus at all ($x^2+y^2+z^2+1=0$) or represent a line ($x^2+y^2=0$).

B. SPHERES. A sphere is the locus of a point (x,y,z) at a given distance r from a given center point (x_0,y_0,z_0). (Fig. 59.)

	EQUATION OF SPHERE	CENTER	RADIUS
1	$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$	(x_0,y_0,z_0)	r
2	$x^2+y^2+z^2+ax+by+cz+d=0$	$(-a/2,-b/2,-c/2)$	$\frac{\sqrt{a^2+b^2+c^2-4d}}{2}$

$a^2+b^2+c^2-4d=0$	locus is degenerate (a point sphere)
$a^2+b^2+c^2-4d<0$	no locus (imaginary sphere)

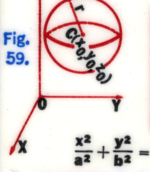

Fig. 59.

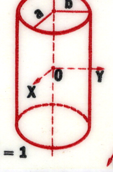

Fig. 60.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

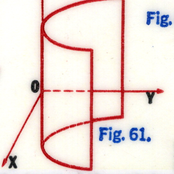

Fig. 61.

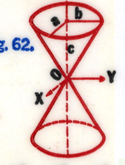

Fig. 62.

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$

C. CYLINDERS. **1.** ***A cylinder*** is the locus of the points on the straight lines that intersect a given plane curve c (other than a straight line) and are perpendicular to the plane of c. **2.** The lines are called ***rulings*** or ***generators*** of the cylinder and ***c*** is its ***directrix.*** When c is a *conic section*, the cylinder is called *quadric cylinder*. **3.** An equation of the 2nd degree in which one variable is absent represents a quadric cylinder whose generators are parallel to the axis corresponding to the absent variable. (Fig. 60, Fig. 61.)

D. CONES. **1.** ***A cone*** is the locus of the points on lines that join the points of a given plane curve c (not a straight line) to a fixed point O not in plane of c. **2.** O is ***vertex***, ***c*** the ***directrix***, and lines joining O to c are the ***rulings*** or ***generators.*** The two parts of the surface that meet at O are called the ***nappes.*** **3.** ***A quadric cone*** is one whose directrix is a proper conic section. A quadric cone whose vertex is at the origin is represented by a homogeneous equation of the 2nd degree: $ax^2+by^2+cz^2+dxy+eyz+fxz=0$ (Fig. 62.)

E. SURFACES OF REVOLUTION. **1.** When a plane curve is rotated about an axis line L in the plane of the curve, a surface of revolution is generated. **2.** Spheres, circular cylinders, circular cones, and surfaces generated by rotating conics about their axes, are quadric surfaces of revolution. **3.** The equation of the surface generated by revolving a curve about a coordinate axis in its plane, is as follows:

REV.	EQ.	REPLACE		REV.	EQ.	REPLACE		REV.	EQ.	REPLACE	
Z-	$f(y,z)=0$	$\lvert y\rvert$	by	Y-	$f(x,y)=0$	$\lvert x\rvert$	by	X-	$f(x,y)=0$	$\lvert y\rvert$	by
AXIS	$f(x,z)=0$	$\lvert x\rvert$	$\sqrt{x^2+y^2}$	AXIS	$f(y,z)=0$	$\lvert z\rvert$	$\sqrt{x^2+z^2}$	AXIS	$f(x,z)=0$	$\lvert z\rvert$	$\sqrt{y^2+z^2}$

E: Find the equation of the surface generated when the ellipse $x^2+2y^2=1$ is rotated around the y-axis. **Sol.:** Substitute x^2+z^2 for x^2; the surface of revolution is $x^2+z^2+2y^2=1$.

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